GEOMETRY – SHEET 4 – Isometries, Change of Coordinates.

1. (i) Let
$$A = \begin{pmatrix} 1 & a & b \\ c & d & -1 \\ e & \frac{1}{2} & f \end{pmatrix}.$$
Are there constants a, b, c, d, e, f such that A is orthogonal?

(ii) If an orthogonal matrix represents a reflection, show that it is symmetric. Is the converse true?

(iii) Let A be an $n \times n$ matrix for which there exists an orthogonal matrix P such that $P^T AP$ is diagonal. Show that A is symmetric.

2. (i) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the map given by
$$T(\mathbf{x}) = A\mathbf{x} + \mathbf{b} \quad \text{where} \quad A = \begin{pmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$
Show that $A^T A = I_2$ and deduce that T is an isometry. What is the image of T?

(ii) Show that there is no isometry from $\mathbb{R}^3$ to $\mathbb{R}^2$.

3. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be an orthonormal basis in $\mathbb{R}^3$ which is right-handed (so that $\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_3$, $\mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_1$, $\mathbf{e}_3 \wedge \mathbf{e}_1 = \mathbf{e}_2$). Say that
$$X\mathbf{e}_1 + Y\mathbf{e}_2 + Z\mathbf{e}_3 = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \qquad \text{and} \qquad \tilde{X}\mathbf{e}_1 + \tilde{Y}\mathbf{e}_2 + \tilde{Z}\mathbf{e}_3 = \tilde{x}\mathbf{i} + \tilde{y}\mathbf{j} + \tilde{z}\mathbf{k}.$$
Show that
$$X\tilde{X} + Y\tilde{Y} + Z\tilde{Z} = x\tilde{x} + y\tilde{y} + z\tilde{z}$$
and
$$(Y\tilde{Z} - Z\tilde{Y})\mathbf{e}_1 + (Z\tilde{X} - X\tilde{Z})\mathbf{e}_2 + (X\tilde{Y} - Y\tilde{X})\mathbf{e}_3 = (y\tilde{z} - z\tilde{y})\mathbf{i} + (z\tilde{x} - x\tilde{z})\mathbf{j} + (x\tilde{y} - y\tilde{x})\mathbf{k}.$$
What is the significance of these identities?

4. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^3$. What does it mean to say that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ forms a basis for $\mathbb{R}^3$?

(i) Suppose a vector $\mathbf{x} \in \mathbb{R}^3$ has coordinates (x_1, x_2, x_3) in the basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. Give a formula for the length of $\mathbf{x}$ in terms of its coordinates and properties of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. What properties of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are thus required in order for the *usual* length formula to hold?

Hence or otherwise, determine the equation of the unit sphere in the coordinate system defined by the basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

(ii) Suppose $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ and $\{\mathbf{U}, \mathbf{V}, \mathbf{W}\}$ are both *orthonormal* bases in $\mathbb{R}^3$, with coordinates given by $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{X} = (X_1, X_2, X_3)$, respectively. By considering the transformation of coordinates in converting from one basis to the other, show that the equation of the unit sphere is invariant.

5. Let $\mathbf{n}$ be a unit vector in $\mathbb{R}^3$. The *stretch* S_k with invariant plane $\mathbf{r} \cdot \mathbf{n} = c$ and stretch factor k is defined by
$$S_k(\mathbf{v}) = \mathbf{v} + (k-1)\,(\mathbf{v} \cdot \mathbf{n} - c)\,\mathbf{n}.$$

(i) Describe the maps S_1, S_0 and S_{-1}.

(ii) Determine the matrix for S_k when the invariant plane is $x + y + z = 0$.

(iii) Find an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of $\mathbb{R}^3$ such that $\mathbf{e}_1, \mathbf{e}_2$ are parallel to the plane in (ii). Describe the map S_k in terms of the co-ordinates associated with $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

6. (*Optional*) Consider the system of differential equations
$$\begin{pmatrix} x_1''(t) \\ x_2''(t) \\ x_3''(t) \end{pmatrix} = \frac{g}{ml} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -5 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}, \tag{1}$$

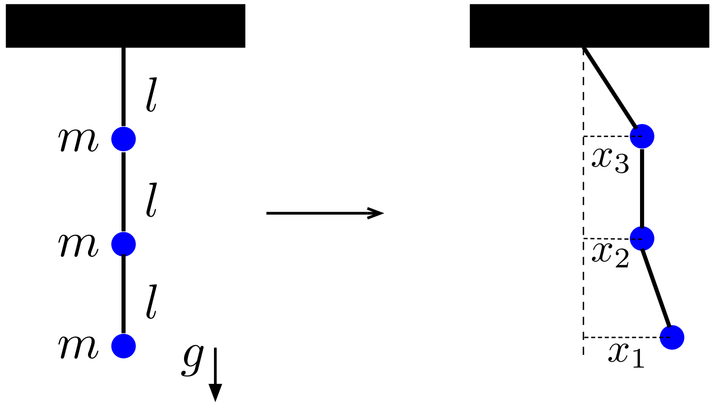

Figure 1: The triple pendulum system

which we could abbreviate $\mathbf{x}''(t) = \frac{g}{ml} A\mathbf{x}(t)$. These equations provide a simple model of the dynamics of a 'triple pendulum', a system of 3 equal masses m hanging under the force of gravity g and connected by strings of length l (see the figure below). The variables $x_i(t)$ denote the horizontal displacement.

Noting that A is symmetric, Spectral Theory gives us that A can be expressed as $A = PDP^T$ for an orthogonal matrix P and diagonal matrix

$$D = \begin{pmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_3 \end{pmatrix}$$

and in this case the $\lambda_i > 0$. Use this fact to determine the general solution for $\mathbf{x}(t)$. [You need not explicitly compute P or the λ_i.]